MATH 1010 E Week 12 Lecture Notes (Martin Li)

Last time ... Integration by Part : $\int u\,dv = uv - \int v\,du$

Trigonometric integrals : $\int \sin^2 x\,dx$ e.g.

Identities :
$$\begin{cases} \cos^2 x + \sin^2 x = 1 \\ 1 + \tan^2 x = \sec^2 x . \\ 1 + \cot^2 x = \csc^2 x . \end{cases}$$

Trigonometric Substitutions

Guiding Example : Consider the integral

$$\int \sqrt{1-x^2}\,dx .$$

Ex: Do integration by part!

Do substitution, let $x = \sin\theta$. then

$$\begin{cases} \sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = \cos\theta . \\ dx = d(\sin\theta) = \cos\theta\,d\theta . \end{cases}$$

$$\Rightarrow \int \sqrt{1-x^2}\,dx = \int \cos\theta \cdot \cos\theta\,d\theta = \int \cos^2\theta\,d\theta .$$

$$= \frac{1}{2}\int (1+\cos 2\theta)\,d\theta = \frac{1}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) + C$$

$$= \frac{1}{2}\left(\sin^{-1}x + x\sqrt{1-x^2}\right) + C \quad \#$$

$\sin 2\theta = 2\sin\theta\cos\theta$

$= 2\sin\theta\sqrt{1-\sin^2\theta}$

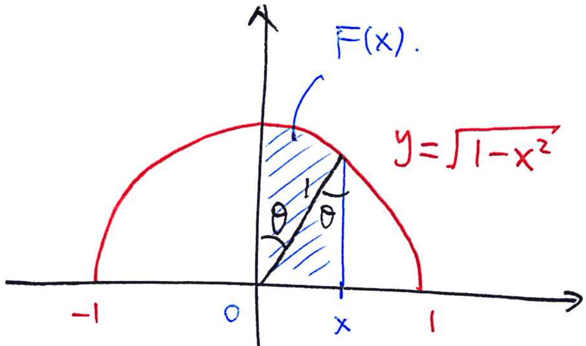
$= 2x\sqrt{1-x^2}$

Note: This is a more geometric solution.

Fundamental Thm I $\Rightarrow$ $F(x) := \int_0^x \sqrt{1-t^2}\,dt$

Q: What is this geometrically?

then $F'(x) = \sqrt{1-x^2}$.



$\boxed{x = \sin\theta}$

Area () $= \frac{1}{2}\theta = \frac{1}{2}\sin^{-1}x$

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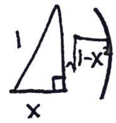
Area () $= \frac{1}{2}x\sqrt{1-x^2}$

$$F(x) = \frac{1}{2}\left(\sin^{-1}x + x\sqrt{1-x^2}\right)$$

Theorems: (1) $\displaystyle\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C$.

Let $a>0$ be a constant.

(2) $\displaystyle\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

(3) $\displaystyle\int \frac{1}{x\sqrt{x^2-a^2}}\,dx = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + C$.

Proof: (1) Take $x = a\sin\theta$, then $dx = a\cos\theta\,d\theta$.

and $\displaystyle\frac{1}{\sqrt{a^2-x^2}} = \frac{1}{\sqrt{a^2-a^2\sin^2\theta}} = \frac{1}{a\cos\theta}$.

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \int \frac{1}{a\cos\theta}\cdot a\cos\theta\,d\theta = \theta + C$$

$$= \sin^{-1}\left(\frac{x}{a}\right) + C \quad \#.$$

(2) Take $x = a\tan\theta$, $dx = a\sec^2\theta\,d\theta$.

and $\displaystyle\frac{1}{a^2+x^2} = \frac{1}{a^2+a^2\tan^2\theta} = \frac{1}{a^2\sec^2\theta}$

$$\int \frac{1}{a^2+x^2}\,dx = \int \frac{1}{a^2\sec^2\theta}\cdot a\sec^2\theta\,d\theta = \frac{1}{a}\theta + C$$

$$= \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C \quad \#$$

(3) Take $x = a\sec\theta$, then $dx = a\sec\theta\tan\theta\, d\theta$.

$$\frac{1}{x\sqrt{x^2-a^2}} = \frac{1}{a\sec\theta\sqrt{a^2\sec^2\theta - a^2}} = \frac{1}{a\sec\theta \cdot a\tan\theta}$$

$$\int \frac{1}{x\sqrt{x^2-a^2}}dx = \int \frac{1}{a^{\cancel{2}}\cancel{\sec\theta}\cancel{\tan\theta}} \cdot \cancel{a}\cancel{\sec\theta}\cancel{\tan\theta}\, d\theta$$

$$= \frac{\theta}{a} + C = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + C \quad \#.$$

Summary:

$\sqrt{a^2-x^2} \longleftrightarrow x = a\sin\theta$ or $a\cos\theta$.

$a^2+x^2 \longleftrightarrow x = a\tan\theta$ or $a\cot\theta$.

$\sqrt{x^2-a^2} \longleftrightarrow x = a\sec\theta$ or $a\csc\theta$.

E.g.: (a) $\displaystyle\int \frac{x^3}{\sqrt{4-x^2}}dx = \int \frac{8\sin^3\theta}{\cancel{2\cos\theta}} \cdot \cancel{2\cos\theta}\, d\theta$

take $x = 2\sin\theta$

$dx = 2\cos\theta\, d\theta$

$$= \int 8\sin^3\theta\, d\theta.$$

$\sin\theta = \frac{x}{2}$

$\cos\theta = \sqrt{1-\frac{x^2}{4}}$

$$= 8\int \sin^2\theta(\sin\theta\, d\theta)$$

$$= 8\int \sin^2\theta\,(-d(\cos\theta))$$

$$= -8\int (1-\cos^2\theta)\, d(\cos\theta)$$

$$= -8\left(\cos\theta - \frac{\cos^3\theta}{3}\right) + C$$

$$= -8\left(1 - \frac{\cos^2\theta}{3}\right)\cos\theta + C$$

$$= -8\left(1 - \frac{1}{3}\left(1-\frac{x^2}{4}\right)\right)\sqrt{1-\frac{x^2}{4}} + C \quad \#.$$

Ex: $\sqrt{a^2-x^2}$

$= \sqrt{a^2 - a^2\sin^2\theta}$

$= \sqrt{a^2\cos^2\theta}$

$= a|\cos\theta|$

(b) $\displaystyle\int \sqrt{\frac{1+x}{1-x}}\,dx = \int \sqrt{\frac{(1+x)^2}{1-x^2}}\,dx$

$$= \int \frac{1+x}{\sqrt{1-x^2}}\,dx .$$

$$= \int \frac{1}{\sqrt{1-x^2}}\,dx + \int \frac{x}{\sqrt{1-x^2}}\,dx$$

$$= \sin^{-1}x + \left(-\frac{1}{2}\right)\int \frac{d(1-x^2)}{\sqrt{1-x^2}} .$$

$$= \sin^{-1}x - \frac{1}{2}\,\frac{\sqrt{1-x^2}}{1/2} + C$$

$$= \sin^{-1}x - \sqrt{1-x^2} + C \quad ※.$$

(c) $\displaystyle\int \frac{dx}{\sqrt{4+x^2}} = \int \frac{\cancel{2}\sec^{\cancel{2}}\theta\,d\theta}{\cancel{2}\,\cancel{\sec\theta}} = \int \sec\theta\,d\theta$

$x = 2\tan\theta$

$dx = 2\sec^2\theta\,d\theta$

$$= \ln|\sec\theta + \tan\theta| + C$$

$$= \ln\left|\sqrt{1+\frac{x^2}{4}} + \frac{x}{2}\right| + C \quad ※.$$

$\tan\theta = \frac{x}{2}$

$\sec\theta = \sqrt{1+\tan^2\theta}$

$= \sqrt{1+\frac{x^2}{4}}$

Reduction Formula

Question: Calculate $\int \cos^n x \, dx$, $n \geq 1$.

$n=0$: $\int 1 \, dx = x + C$

$n=1$: $\int \cos x \, dx = \sin x + C$.

How do we get a general formula:

$$I_n = \int \cos^n x \, dx \qquad n \geq 1, \; n=0.$$

Transform the integral:

$$\begin{aligned}
I_n = \int \cos^n x \, dx &= \int \cos^{n-1} x \, (\cos x \, dx) \\
&= \int \cos^{n-1} x \, d(\sin x) \\
&= \sin x \cos^{n-1} x - \int \sin x \, d(\cos^{n-1} x) \\
&= \sin x \cos^{n-1} x - \int \sin x \cdot (n-1) \cos^{n-2} x \, \underbrace{(-\sin x)}_{dx} \\
&= \sin x \cos^{n-1} x + (n-1) \int \underbrace{\sin^2 x}_{1-\cos^2 x} \cos^{n-2} x \, dx \\
&= \sin x \cos^{n-1} x + (n-1) \int \cos^{n-2} x - \cos^n x \, dx
\end{aligned}$$

$$I_n = \sin x \cos^{n-1} x + (n-1) I_{n-2} - (n-1) I_n$$

$$\Rightarrow \boxed{I_n = \frac{1}{n} \sin x \cos^{n-1} x + \frac{n-1}{n} I_{n-2}}$$

reduction formula.

$$\Rightarrow \quad I_n \to I_{n-2} \to \cdots \to I_3 \to I_1 = \sin x + C \qquad n \text{ odd}$$

$$I_n \to I_{n-2} \to \cdots \to I_2 \to I_0 = x + C \qquad n \text{ even}$$

Idea: Get any I_n by working backwards.

You get a complicated formula for general I_n.

For definite integrals, it sometimes simplifies.

E.g.
$$\underbrace{\int_0^{\pi/2} \cos^n x\,dx}_{I_n} = \left[\frac{1}{n} \cancel{\sin x \cos^{n-1} x}\right]_{x=0}^{x=\pi/2} + \frac{n-1}{n} \underbrace{\int_0^{\pi/2} \cos^{n-2} x\,dx}_{I_{n-2}.}$$

0.

If n even,

$$I_n = \frac{n-1}{n} I_{n-2} = \frac{n-1}{n} \cdot \frac{n-3}{n-2} I_{n-4}$$

$$= \cdots = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{3}{4} \cdot \frac{1}{2} I_0$$

$$= \frac{(n-1)(n-3)\cdots 3\cdot 1}{n(n-2)\cdots 4\cdot 2} \cdot \left(\frac{\pi}{2}\right). \quad \#$$

Ex: Work out the formula when n is odd.

E.g.:
$$\int_0^{\pi/2} \sin^n x\,dx = ? \int_0^{\pi/2} \cos^n x\,dx.$$

Ex: Get a reduction formula for this.

Note: There is a 1 line proof.

Remember: $\boxed{\sin\left(\frac{\pi}{2} - x\right) = \cos x}$.

Let $x = \frac{\pi}{2} - u$. then $dx = -du$, $\quad x=0 \leftrightarrow u = \frac{\pi}{2}$, $\quad x = \frac{\pi}{2} \leftrightarrow u = 0$

$$\int_0^{\pi/2} \sin^n x\,dx = \int_{\pi/2}^{0} \left[\sin\left(\frac{\pi}{2} - u\right)\right]^n (-du) = \int_0^{\pi/2} \cos^n u\,du$$

Another example: Get a reduction formula for

$$I_n = \int x^n e^{ax}\, dx . \qquad n \geq 0 .$$

$$I_n = \int x^n e^{ax}\, dx = \frac{1}{a}\int x^n\, d(e^{ax})$$

$$= \frac{1}{a} x^n e^{ax} - \frac{1}{a}\int e^{ax}\, d(x^n)$$

$$= \frac{1}{a} x^n e^{ax} - \frac{n}{a}\int x^{n-1} e^{ax}\, dx$$

$$\boxed{I_n = \frac{1}{a} x^n e^{ax} - \frac{n}{a} I_{n-1}}$$

$$I_n \longrightarrow I_{n-1} \longrightarrow I_{n-2} \longrightarrow \cdots \longrightarrow I_3 \longrightarrow I_2 \longrightarrow I_1 \longrightarrow I_0$$

$\uparrow$ base case.

Ex: Derive a reduction formula for

Challenging!

$$I_n = \int \left(\frac{\sin \frac{x-a}{2}}{\sin \frac{x+a}{2}} \right)^n dx \qquad n \geq 1 .$$

Last time ... Reduction formula, Trigonometric substitution.

defined for $x \in (-1,1)$

$$\int \sqrt{1-x^2}\,dx = \int \sqrt{1-\sin^2\theta}\cdot\cos\theta\,d\theta$$

Take $x = \sin\theta$ $\quad = \int \sqrt{\cos^2\theta}\,\cos\theta\,d\theta$

$dx = \cos\theta\,d\theta$

argue that : $\cos\theta > 0$. $\quad = \int |\cos\theta|\cos\theta\,d\theta$

why? : $\theta = \sin^{-1} x$ $\quad = \int \cos^2\theta\,d\theta = \cdots$

$\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\Rightarrow$ $\cos\theta > 0$ in this interval

Piecewise-defined functions

E.g. $\int |x|\,dx = ? = F(x)$ $\quad |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

$$\left.\begin{array}{l} \text{For } x \geq 0, \quad \int |x|\,dx = \int x\,dx = \frac{1}{2}x^2 + C_1 \\ \text{For } x < 0, \quad \int |x|\,dx = \int -x\,dx = -\frac{1}{2}x^2 + C_2 \end{array}\right\} F(x)$$

If I require continuity at $x=0$, ($\because F$ is diff. everywhere)

$$C_1 = \lim_{x\to 0^+} F(x) = \lim_{x\to 0^-} F(x) = C_2$$

$$\Rightarrow \quad \int |x|\,dx = \begin{cases} \frac{1}{2}x^2 + C & \text{if } x \geq 0 \\ -\frac{1}{2}x^2 + C & \text{if } x < 0 \end{cases}$$

Same constant.

Q: What about definite integrals?

① Use Fundamental Thm,

$$\int_{-1}^{1} |x|\,dx = F(1) - F(-1) = \left(\frac{1}{2} + \cancel{c}\right) - \left(-\frac{1}{2} + \cancel{c}\right) = 1 \quad \#$$

cts function.

② Split up the integrals:

$$\int_{-1}^{1} |x|\,dx = \int_{-1}^{0} |x|\,dx + \int_{0}^{1} |x|\,dx$$

$$= \int_{-1}^{0} -x\,dx + \int_{0}^{1} x\,dx$$

$$= -\frac{1}{2}x^2\Big|_{-1}^{0} + \frac{1}{2}x^2\Big|_{0}^{1}$$

$$= \left(0 + \frac{1}{2}\right) + \left(\frac{1}{2} - 0\right) = 1 \quad \#.$$

Eg. 2: Evaluate $\displaystyle\int_0^{\pi/2} \frac{\sin\theta}{\cos\theta + \sin\theta}\,d\theta$.

Idea: Change of variable. : $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$.

Sol: Let $u = \frac{\pi}{2} - \theta$, $du = -d\theta$, $\theta = 0 \iff u = \pi/2$, $\theta = \pi/2 \iff u = 0$.

$$I = \int_0^{\pi/2} \frac{\sin\theta}{\cos\theta + \sin\theta}\,d\theta = \int_{\pi/2}^{0} \frac{\sin\left(\frac{\pi}{2} - u\right)}{\cos\left(\frac{\pi}{2} - u\right) + \sin\left(\frac{\pi}{2} - u\right)}(-du)$$

$$= \int_0^{\pi/2} \frac{\cos u}{\sin u + \cos u}\,du = I$$

$$I + I = \int_0^{\pi/2} \frac{\sin\theta + \cos\theta}{\sin\theta + \cos\theta}\,d\theta = \int_0^{\pi/2} 1\,d\theta = \frac{\pi}{2}$$

$$\Rightarrow \quad I = \frac{\pi}{4} \quad \#.$$

Integration of rational functions (Partial Fraction!)

$\hookrightarrow \dfrac{p(x)}{q(x)}$, p, q polynomials.

A guiding example:

Consider $\displaystyle\int \frac{1}{x(x-1)}\,dx$

If we are fortunate that

$$\frac{1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}$$ where A, B are constants.

If we can do this, then

$$\text{R.H.S.} = \frac{A}{x} + \frac{B}{x-1} = \frac{A(x-1)+Bx}{x(x-1)}$$

match?

Want:

$$\frac{1}{x(x-1)} = \frac{(A+B)x - A}{x(x-1)}$$

$$\Rightarrow \begin{cases} A+B=0 \\ -A=1 \end{cases} \Rightarrow \boxed{\begin{matrix} A=-1 \\ B=1 \end{matrix}}$$

$$\int \frac{1}{x(x-1)}\,dx = \int \left(\frac{-1}{x} + \frac{1}{x-1}\right)dx = -\ln|x| + \ln|x-1| + C$$

\#

Q: When can we do that?

E.g. 3:

$$\frac{x^2-2}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

$$= \frac{A(x-1)^2 + Bx(x-1) + Cx}{x(x-1)^2}$$

$\Rightarrow$ expand & compare coefficients $\Rightarrow$ $\boxed{\begin{matrix} A=-2 \\ B=3 \\ C=-1 \end{matrix}}$

Why not $\dfrac{Cx+D}{(x-1)^2}$

$$\frac{Cx+D}{(x-1)^2} = \frac{C(x-1)+C+D}{(x-1)^2} = \frac{C}{x-1} + \frac{C+D}{(x-1)^2}$$

soo

Eg. 4: $\dfrac{x^2-x+2}{\underbrace{2x^3+3x^2-2x}} = \dfrac{A}{x} + \dfrac{B}{x+2} + \dfrac{C}{2x-1}$

factorize this first:

Ex: Find A, B, C.

Ans: $A=1$, $B=\frac{2}{5}$, $C=-\frac{9}{5}$.

$x(2x^2+3x-2)$

$\parallel$

$x(x+2)(2x-1)$ ← distinct linear factors.

Eg. 5: $\dfrac{2x^3-4x^2-x-3}{\underbrace{x^2-2x-3}_{(x-3)(x+1)}} = 2x + \dfrac{5x-3}{x^2-2x-3}$ ← deg < deg

do long division first:

$$\begin{array}{r} 2x \\ x^2-2x-3 \,\big)\, \overline{2x^3-4x^2-x-3} \\ \underline{2x^3-4x^2-6x} \\ 5x-3 \end{array}$$

Eg. 6: $\dfrac{1}{x(x^2+1)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+1}$

$$= \frac{A(x^2+1)+(Bx+C)x}{x(x^2+1)}$$

$\Rightarrow \quad 1 = \underbrace{(A+B)}_{0}x^2 + \underbrace{C}_{0}x + \underbrace{A}_{1}. \quad \Rightarrow \quad \boxed{\begin{array}{l} A=1 \\ B=-1 \\ C=0 \end{array}}$

$\Rightarrow \quad \displaystyle\int \frac{1}{x(x^2+1)}\,dx = \int\left(\frac{1}{x} - \frac{x}{x^2+1}\right)dx$

$$= \ln|x| - \frac{1}{2}\ln|x^2+1| + C \quad ※.$$